

# Approximation of the finite dimensional distributions of multiple fractional integrals

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## Abstract

We construct a family  $I_{n_\varepsilon}(f)_t$  of continuous stochastic processes that converges in the sense of finite dimensional distributions to a multiple Wiener-Itô integral  $I_n^H(f1_{[0,t]}^{\otimes n})$  with respect to the fractional Brownian motion. We assume that  $H > \frac{1}{2}$  and we prove our approximation result for the integrands  $f$  in a rather general class.

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## 1 Introduction

A first result concerning the approximation of iterated stochastic integrals has been given in [1]. Consider  $(X^\varepsilon)_{\varepsilon>0}$  a family of semimartingales with paths in the Skorohod space  $\mathcal{D}([0, 1])$  that converges weakly in this space to another semimartingale  $X$ , as  $\varepsilon$  tends to zero. It has

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been proven in [1] that the couple  $(X^\varepsilon, [X^\varepsilon, X^\varepsilon])$  converges weakly in  $\mathcal{D}([0, 1])$  as  $\varepsilon \rightarrow 0$  to the couple  $(X, [X, X])$  ( $[X, X]$  denotes the usual semimartingale bracket) if and only if for every  $m \geq 1$  the vector  $(J_1(X^\varepsilon), \dots, J_m(X^\varepsilon))$  converges weakly in  $\mathcal{D}([0, 1])$  as  $\varepsilon \rightarrow 0$  to the vector  $(J_1(X), \dots, J_m(X))$ . Here  $J_1(X)_t = X_t$  and for  $k \geq 2$ ,  $J_k(X)_t = \int_0^t J_{k-1}(X)_{s-} dX_s$  (and similarly for  $J_k(X^\varepsilon)$ ). This result shows that in order to obtain (joint) weak convergence of iterated Itô integrals we need the convergence of  $X^\varepsilon$  to  $X$  but also the convergence of the second order variations. When our semimartingale is the Wiener process, there are many examples of families of processes with absolutely continuous paths converging weakly to it in the topology of  $\mathcal{C}([0, 1])$ . In this case it is obvious that we do not have convergence of the quadratic variations of such families to the quadratic variations of the Brownian motion. This led to the problem of approximating iterated stochastic integrals with respect to the Brownian motion and later, with respect to the fractional Brownian motion.

Let us recall some relatively recent results concerning the approximation of iterated integrals with respect to a standard Brownian motion by a family of processes with continuous paths. Consider a family of stochastic processes  $(\rho_\varepsilon)_{\varepsilon>0}$  of the form

$$\rho_\varepsilon(t) = \int_0^t \theta_\varepsilon(s) ds$$

such that  $(\rho_\varepsilon)_{\varepsilon>0}$  converges weakly in  $C_0([0, 1])$  (the space of continuous function on  $[0, 1]$  which are null at zero) to the Wiener process. We will discuss two main examples: the case when  $\theta_\varepsilon(s) := \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \xi_k I_{[k-1, k)}\left(\frac{s}{\varepsilon^2}\right)$ , where  $\{\xi_k\}$  is a sequence of independent, identically distributed random variables satisfying  $E(\xi_1) = 0$  and  $\text{Var}(\xi_1) = 1$  (these kernels are traditionally called *Donsker kernels*) and the case when

$$\theta_\varepsilon(x) := \frac{1}{\varepsilon} (-1)^{N\left(\frac{x}{\varepsilon^2}\right)},$$

where  $N = \{N(s); s \geq 0\}$  is a standard Poisson process (these kernels are usually called *Stroock kernels* or *Kac-Stroock kernels* because they were introduced by Kac in [6] and used by Stroock, [10], in order to obtain weak approximations of the Brownian motion). In [2] the authors proved that, for a suitable function  $f$  defined on  $[0, 1]^{\otimes n}$ , the family of multiple integrals  $(I_{n_\varepsilon}^1(f))_{\varepsilon>0}$  with respect to  $\rho_\varepsilon$  given by

$$\begin{aligned} I_{n_\varepsilon}^1(f)_t &= \int_{[0, t]^n} f(t_1, \dots, t_n) d\rho_\varepsilon(t_1) \cdots d\rho_\varepsilon(t_n) \\ &= \int_{[0, t]^n} f(t_1, \dots, t_n) \theta_\varepsilon(t_1) \cdots \theta_\varepsilon(t_n) dt_1 \cdots dt_n \end{aligned} \quad (1)$$

converges weakly in  $C_0([0, 1])$  to the  $n$ th multiple Stratonovich integral of  $f 1_{[0, t]}^{\otimes n}$  with respect to the standard Brownian motion. This is somehow expected because the Stratonovich integral usually satisfies the differential rules of the deterministic calculus. In order to obtain as a limit a multiple Itô integral (which has zero expectation) one needs to subtract

the “trace” of  $I_{n_\varepsilon}^1(f)$ , that means, to suppress the values on the diagonals. This following result has been obtained in [4]: for any  $f \in L^2([0, 1]^n)$ , the family  $(I_{n_\varepsilon}^2(f))_{\varepsilon > 0}$  given by

$$\begin{aligned} I_{n_\varepsilon}^2(f)_t &= \int_{[0, t]^n} f(x_1, x_2, \dots, x_n) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} d\rho_\varepsilon(x_1) \cdots d\rho_\varepsilon(x_n) \\ &= \int_{[0, t]^n} f(x_1, x_2, \dots, x_n) \prod_{i=1}^n \theta_\varepsilon(x_i) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} dx_1 \cdots dx_n, \end{aligned} \quad (2)$$

converges weakly, in the sense of finite dimensional distributions (and in  $C_0([0, 1])$  for  $n = 2$ ), to the  $n$ th multiple Itô integral  $I_n(f 1_{[0, t]}^{\otimes n})$ . Let us consider now the problem of approximating the fractional Brownian motion  $(B_t^H)_{t \in [0, 1]}$  and the multiple integrals with respect to it. Recall that the fractional Brownian motion is a centered Gaussian process with covariance  $R(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$  with  $H \in (0, 1)$ . It can be also expressed as a Wiener integral with respect to a Wiener process  $W$  by  $B^H = \int_0^t K_H(t, s) dW_s$  where  $K_H$  is a deterministic kernel defined on the set  $\{0 < s < t\}$  and given by

$$K_H(t, s) = c_H(t - s)^{H - \frac{1}{2}} + c_H \left( \frac{1}{2} - H \right) \int_s^t (u - s)^{H - \frac{3}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2} - H} \right) du, \quad (3)$$

where  $c_H$  is the normalizing constant  $c_H = \left( \frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \right)^{\frac{1}{2}}$ . From this representation and the weak convergence of  $\rho_\varepsilon$  to  $W$  it follows that (see [3]) for any  $H \in (0, 1)$  the family of processes  $(\eta_\varepsilon)_{\varepsilon > 0}$  with

$$\eta_\varepsilon(t) = \int_0^t K_H(t, s) \theta_\varepsilon(s) ds, \quad t \in [0, 1]$$

converges weakly as  $\varepsilon \rightarrow 0$  in  $C_0([0, 1])$  to  $B^H$ . When  $H > \frac{1}{2}$  the paths of  $\eta_\varepsilon$  are even absolutely continuous. Moreover, if  $H > \frac{1}{2}$ , the multiple integral with respect to  $\eta_\varepsilon$

$$I_{n_\varepsilon}^3(f)_t = \int_{[0, t]^{\otimes n}} f(t_1, \dots, t_n) d\eta_\varepsilon(t_1) \cdots d\eta_\varepsilon(t_n) \quad (4)$$

converges as  $\varepsilon \rightarrow 0$  in  $C_0([0, 1])$  to the multiple Stratonovich integral of order  $n$  of the function  $f 1_{[0, t]}^{\otimes n}$  with respect to  $B^H$ . The purpose of this work is to give an approximation result for the multiple Wiener-Itô integrals  $I_n^H(f 1_{[0, t]}^{\otimes n})$  with respect to the fractional Brownian motion, for the integrand  $f$  in a rather general class of functions. Note that, as we recall in Section 2, the multiple fractional integral  $I_n^H$  can be expressed as a multiple Wiener-Itô integral with respect to the Brownian motion. In fact, we have  $I_n^H(f 1_{[0, t]}^{\otimes n}) = I_n \left( \Gamma_H^{(n)} f 1_{[0, t]}^{\otimes n} \right)$

where  $\Gamma_H^{(n)}$  is a transfer operator. Concretely, we show here that the family  $(I_{n_\varepsilon}(f))_{\varepsilon>0}$  defined by

$$I_{n_\varepsilon}(f)_t = \int_0^1 \dots \int_0^1 \left( \Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n} \right) (x_1, \dots, x_n) \left( \prod_{i=1}^n \theta_\varepsilon(x_i) \right) \prod_{i,j=1; i \neq j}^n 1_{\{|x_i - x_j| > \varepsilon\}} dx_1 \dots dx_n$$

converges, in the sense of finite dimensional distributions, to  $(I_n^H(f 1_{[0,t]}^{\otimes n}))_{t \in [0,1]}$ . Due to the rather complicate expression of the operator  $\Gamma_H^{(n)}$  this result cannot be deduced from the result in [4] since the transfer principle for multiple fractional integrals actually implies that  $I_n^H(f 1_{[0,t]}^{\otimes n})$  is equal to  $I_n(g(t, \cdot) 1_{[0,t]}^{\otimes n})$  with some function  $g$  depending on  $f$ . Because of the appearance of the variable  $t$  in the argument of  $g$ , the main result in [4] cannot be directly applied. Another particularity of the multiple fractional integrals is that the expectation  $E I_1^H(1_A) I_1^H(1_B)$  is not zero when  $A$  and  $B$  are disjoint subsets of  $[0,1]$  and this fact makes the proofs considerably more complex than in the standard Brownian motion case.

We structured our paper in the following way. Section 2 contains some preliminaries on multiple Wiener-Itô integrals and multiple integrals with respect to the fractional Brownian motion. In Section 3 we prove our approximation result. We first regard the case when the integrand is a step function. We separated the case  $n = 2$  and  $n \geq 3$  because in the first case the proof is less complex and more intuitive and it helps to understand the general case. Finally we extend our result from simple functions to a bigger class of functions.

## 2 Preliminaries

### 2.1 Multiple Wiener-Itô integrals

In this paragraph we describe the basic elements of calculus on Wiener chaos. Let  $(W_t)_{t \in [0,1]}$  be a classical Wiener process on a standard Wiener space  $(\Omega, \mathcal{F}, \mathbf{P})$ . If  $f \in L^2([0,1]^n)$  with  $n \geq 1$  integer, we introduce the multiple Wiener-Itô integral of  $f$  with respect to  $W$ . We refer to [7] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals.

Let  $f \in \mathcal{S}_m$  be an elementary functions with  $m$  variables that can be written as

$$f = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} 1_{A_{i_1} \times \dots \times A_{i_m}}$$

where the coefficients satisfy  $c_{i_1, \dots, i_m} = 0$  if two indices  $i_k$  and  $i_l$  are equal and the sets  $A_i \in \mathcal{B}([0,1])$  are pairwise-disjoint. For a such step function  $f$  we define

$$I_m(f) = \sum_{i_1, \dots, i_m} c_{i_1, \dots, i_m} W(A_{i_1}) \dots W(A_{i_m})$$

where we put  $W([a, b]) = W_b - W_a$ . It can be seen that the application  $I_m$  constructed above from  $\mathcal{S}_m$  to  $L^2(\Omega)$  is an isometry on  $\mathcal{S}_m$ , i.e.

$$E[I_n(f)I_m(g)] = n!\langle f, g \rangle_{L^2([0,1]^n)} \text{ if } m = n \quad (5)$$

and

$$E[I_n(f)I_m(g)] = 0 \text{ if } m \neq n.$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where  $\tilde{f}$  denotes the symmetrization of  $f$  defined by  $\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ .

Since the set  $\mathcal{S}_n$  is dense in  $L^2([0, 1]^n)$  for every  $n \geq 1$  the mapping  $I_n$  can be extended to an isometry from  $L^2([0, 1]^n)$  to  $L^2(\Omega)$  and the above properties hold true for this extension. Note also that  $I_n$  can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n};$$

here the integrals are of Itô type; this formula is easy to show for elementary functions  $f$ , and follows for general  $f \in L^2([0, 1]^n)$  by a density argument.

The product for two multiple integrals says that (see [7]): if  $f \in L^2([0, 1]^n)$  and  $g \in L^2([0, 1]^m)$  are symmetric functions, then

$$I_n(f)I_m(g) = \sum_{l=0}^{m \wedge n} l! \binom{m}{l} \binom{n}{l} I_{m+n-2l}(f \otimes_l g) \quad (6)$$

where the contraction  $f \otimes_l g$  belongs to  $L^2([0, 1]^{m+n-2l})$  for  $l = 0, 1, \dots, m \wedge n$  and it is given by

$$\begin{aligned} & (f \otimes_l g)(s_1, \dots, s_{n-l}, t_1, \dots, t_{m-l}) \\ &= \int_{[0,1]^l} f(s_1, \dots, s_{n-l}, u_1, \dots, u_l) g(t_1, \dots, t_{m-l}, u_1, \dots, u_l) du_1 \dots du_l. \end{aligned} \quad (7)$$

When  $l = 0$ , we will denote, throughout this paper, by  $f \otimes g := f \otimes_0 g$ .

## 2.2 Multiple fractional integrals

Let us introduce here the multiple integrals with respect to the fractional Brownian motion. We follow the approach in [8] (see also [7] and [9]). Let  $f \in L^1([0, 1]^n)$  and for every  $0 < \alpha < 1$  define the operator

$$(I_{t-}^{\alpha, n} f)(x_1, \dots, x_n) = \frac{1}{(\Gamma(\alpha))^n} \int_{x_1}^t \dots \int_{x_n}^t \frac{f(t_1, \dots, t_n)}{\prod_{j=1}^n (t_j - x_j)^{1-\alpha}} dt_1 \dots dt_n$$

for every  $x_1, \dots, x_n \in [0, t]$  with  $t \in [0, 1]$ .

We have the following properties:

- if  $f = f_1 \otimes \cdots \otimes f_n$  with  $f_i \in L^1([0, 1])$  then

$$(I_{t-}^{\alpha, n} f)(x_1, \dots, x_n) = (I_{t-}^{\alpha, 1} f_1)(x_1) \cdots (I_{t-}^{\alpha, 1} f_n)(x_n)$$

for every  $x_1, \dots, x_n \in [0, t]$ .

- If  $H > \frac{1}{2}$  then

$$c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} \left( I_{1-}^{H-\frac{1}{2}, 1} (x^{H-\frac{1}{2}} 1_{[0, t]}) \right) (s) = K_H(t, s) \quad (8)$$

where  $K_H$  is the standard kernel of the fractional Brownian motion (3).

We introduce the space  $|\mathcal{H}|^{\otimes n}$  of measurable functions  $f : [0, 1]^n \rightarrow \mathbb{R}$  such that

$$\int_{[0, 1]^{2n}} |f(u_1, \dots, u_n) f(v_1, \dots, v_n)| \left( \prod_{j=1}^n \psi(u_j, v_j) \right) du_1 \cdots du_n dv_1 \cdots dv_n < \infty$$

where  $\psi(s, t) = H(2H - 1)|s - t|^{2H-2}$ .

**Remark 1** For any  $H > \frac{1}{2}$  we have (see [7], [8])

$$L^2([0, 1]^n) \subset L^{\frac{1}{H}}([0, 1]^n) \subset |\mathcal{H}|^{\otimes n}.$$

Define the operator  $\Gamma_H^{(n)} : |\mathcal{H}|^{\otimes n} \rightarrow L^2([0, 1]^n)$

$$(\Gamma_H^{(n)} f)(t_1, \dots, t_n) = \left[ c_H \Gamma \left( H + \frac{1}{2} \right) \right]^n \prod_{j=1}^n t_j^{\frac{1}{2}-H} \left( I_{1-}^{H-\frac{1}{2}, n} \right) (f(x_1, \dots, x_n) \prod_{j=1}^n x_j^{H-\frac{1}{2}})(t_1, \dots, t_n) \quad (9)$$

Then the operator  $\Gamma_H^{(n)}$  is an isometry between  $|\mathcal{H}|^{\otimes n}$  and  $L^2([0, 1]^n)$  where we endow the space  $|\mathcal{H}|^{\otimes n}$  with the following inner product

$$\langle f, g \rangle_{\mathcal{H}^{\otimes n}} = \int_{[0, 1]^{2n}} f(u_1, \dots, u_n) g(v_1, \dots, v_n) \left( \prod_{j=1}^n \psi(u_j, v_j) \right) du_1 \cdots du_n dv_1 \cdots dv_n.$$

Note that

- if  $f = 1_{[0, b]}$  then by (8)

$$(\Gamma_H^{(1)} 1_{[0, b]})(s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} I_{1-}^{H-\frac{1}{2}} (x^{H-\frac{1}{2}} 1_{[0, b]})(s) = K_H(b, s)$$

- if  $f = 1_{(a, b]}$  then  $(\Gamma_H^{(1)} 1_{(a, b]})(s) = K_H(b, s) - K_H(a, s)$ .

- If  $f_i \in |\mathcal{H}|$  ( $i = 1, \dots, n$ ) then

$$\Gamma_H^{(n)}(f_1 \otimes \dots \otimes f_n) = \Gamma_H^{(1)} f_1 \otimes \dots \otimes \Gamma_H^{(1)} f_n. \quad (10)$$

Let  $f \in |\mathcal{H}|^{\otimes n}$ . Then we define the multiple Wiener-Itô integral of  $f$  with respect to the fractional Brownian motion by

$$I_n^H(f) = I_n(\Gamma^{(n)} f) \quad (11)$$

where  $I_n$  denotes the standard Wiener-Itô integral with respect to the Wiener process as defined above. Note that  $\Gamma^{(n)} f \in L^2([0, 1]^n)$ .

### 3 Approximation of multiple fractional Wiener integrals

Let us introduce some notation. We set

$$\eta_\varepsilon(t) = \int_0^t K_H(t, s) \theta_\varepsilon(s) ds, \quad t \in [0, 1] \quad (12)$$

where  $\theta_\varepsilon$  is such that  $\int_0^t \theta_\varepsilon(s) ds$  converges weakly in the topology of the space  $\mathcal{C}_0([0, 1])$  to the standard Brownian motion.

**Lemma 1** *Let  $\theta_\varepsilon$  be either the Kac-Stroock kernels or the Donsker kernels. Then the family of processes  $\eta_\varepsilon$  converges weakly in  $\mathcal{C}_0([0, 1])$  as  $\varepsilon \rightarrow 0$  to the fractional Brownian motion  $B^H$  for any  $H \in (0, 1)$ .*

**Proof:** It has been proved in [3], Proposition 2.1. ■

Denote, for every  $\varepsilon > 0$

$$g_\varepsilon(x_1, \dots, x_n) = \prod_{i,j=1; i \neq j}^n 1_{\{|x_i - x_j| > \varepsilon\}} \quad (13)$$

and

$$I_{n_\varepsilon}(f)_t = \int_0^1 \dots \int_0^1 \left( \Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n} \right) (x_1, \dots, x_n) \left( \prod_{i=1}^n \theta_\varepsilon(x_i) \right) g_\varepsilon(x_1, \dots, x_n) dx_1 \dots dx_n. \quad (14)$$

**Remark 2** *Note that it follows from a result in [9] that, if  $f \in L^q([0, 1])$  for some  $q > \frac{1}{H}$*

then the function  $t \rightarrow I_{n_\varepsilon}(f)_t$  is continuous. Indeed, for every  $s < t$

$$\begin{aligned}
& |I_{n_\varepsilon}(f)_t - I_{n_\varepsilon}(f)_s| \\
& \leq \sup_{0 \leq r \leq 1} |\theta_\varepsilon(r)|^n \int_{[0,1]^{\otimes n}} \left| (\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n})(x_1, \dots, x_n) - (\Gamma_H^{(n)} f 1_{[0,s]}^{\otimes n})(x_1, \dots, x_n) \right| dx_1 \cdots dx_n \\
& \leq \sup_{0 \leq r \leq 1} |\theta_\varepsilon(r)|^n \left( \int_{[0,1]^{\otimes n}} \left| (\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n})(x_1, \dots, x_n) - (\Gamma_H^{(n)} f 1_{[0,s]}^{\otimes n})(x_1, \dots, x_n) \right|^2 dx_1 \cdots dx_n \right)^{\frac{1}{2}} \\
& = \sup_{0 \leq r \leq 1} |\theta_\varepsilon(r)|^n \left( E \left| I_n^H(f 1_{[0,t]}^{\otimes n}) - I_n^H(f 1_{[0,s]}^{\otimes n}) \right|^2 \right)^{\frac{1}{2}} \\
& \leq C_{H,n} \sup_{0 \leq r \leq 1} |\theta_\varepsilon(r)|^n |t - s|^{H - \frac{1}{q}},
\end{aligned}$$

where for the last inequality we used Theorem 3.2 in [9].

We first prove the following result.

**Lemma 2** *Let  $f$  be a simple function of the form*

$$f(x_1, \dots, x_n) = \sum_{k=1}^m \alpha_k 1_{\Delta_k}(x_1, \dots, x_n) \quad (15)$$

where  $m \in \mathbb{N}$ ,  $\alpha_k \in \mathbb{R}$  for every  $k = 1, \dots, m$  and  $\Delta_k = (a_k^1, b_k^1] \times \cdots \times (a_k^n, b_k^n]$  such that for every  $k = 1, \dots, m$ ,  $(a_k^i, b_k^i]$  are disjoint intervals ( $i = 1, \dots, n$ ). Then the finite dimensional distributions of the process  $(Y^\varepsilon(f)_t)_{t \in [0,1]}$  given by

$$Y^\varepsilon(f)_t := \int_0^1 \cdots \int_0^1 \left( \Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n} \right)(x_1, \dots, x_n) \left( \prod_{i=1}^n \theta_\varepsilon(x_i) \right) dx_1 \cdots dx_n,$$

converge as  $\varepsilon \rightarrow 0$  to the finite dimensional distributions of

$$\begin{aligned}
& \left( \sum_{k=1}^m \alpha_k I_1^H(1_{(a_k^1, b_k^1]} 1_{[0,t]}) \cdots I_1^H(1_{(a_k^n, b_k^n]} 1_{[0,t]}) \right)_{t \in [0,1]} \\
& = \left( \sum_{k=1}^m \alpha_k \left( B_{b_k^1 \wedge t}^H - B_{a_k^1 \wedge t}^H \right) \cdots \left( B_{b_k^n \wedge t}^H - B_{a_k^n \wedge t}^H \right) \right)_{t \in [0,1]}.
\end{aligned}$$



**Proof:** We have, by using the property (10) of the operator  $\Gamma_H^{(n)}$ ,

$$\begin{aligned}
Y^\varepsilon(f)_t &= \int_0^1 \dots \int_0^1 \left( \Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n} \right) (x_1, \dots, x_n) \left( \prod_{i=1}^n \theta_\varepsilon(x_i) \right) dx_1 \dots dx_n \\
&= \sum_{k=1}^m \alpha_k \int_0^1 \dots \int_0^1 \left( \Gamma_H^{(n)} 1_{(a_k^1, b_k^1] \times \dots \times (a_k^n, b_k^n]} 1_{[0,t]}^{\otimes n} \right) (x_1, \dots, x_n) \left( \prod_{i=1}^n \theta_\varepsilon(x_i) \right) dx_1 \dots dx_n \\
&= \sum_{k=1}^m \alpha_k \prod_{i=1}^n \int_0^1 \left( \Gamma_H^{(1)} 1_{(a_k^i, b_k^i]} 1_{[0,t]} \right) (x_i) \theta_\varepsilon(x_i) dx_i \\
&= \sum_{k=1}^m \alpha_k \prod_{i=1}^n \int_0^1 \left( K_H(b_k^i \wedge t, x_i) - K_H(a_k^i \wedge t, x_i) \right) \theta_\varepsilon(x_i) dx_i \\
&= \sum_{k=1}^m \alpha_k \prod_{i=1}^n (\eta_\varepsilon(b_k^i \wedge t) - \eta_\varepsilon(a_k^i \wedge t)).
\end{aligned}$$

and then for every fixed  $t_1, \dots, t_r \in [0, 1]$  the vector  $(Y^\varepsilon(f)_{t_1}, \dots, Y^\varepsilon(f)_{t_r})$  converges as in the statement because by Lemma 1  $\eta_\varepsilon$  converges weakly to the fractional Brownian motion.

■

**Remark 3** Let  $f$  be a simple function. It can be seen that (here  $\partial_1 K_H$  denotes the partial derivative of  $K_H$  with respect to the first variable)

$$\begin{aligned}
&Y^\varepsilon(f)_t \\
&= \int_{[0,t]^n} dx_1 \dots dx_n \left( \int_{x_1}^t \dots \int_{x_n}^t \partial_1 K_H(t_1, x_1) \dots \partial_1 K_H(t_n, x_n) f(t_1, \dots, t_n) dt_1 \dots dt_n \right) \left( \prod_{i=1}^n \theta_\varepsilon(x_i) \right) \\
&= \int_{[0,t]^n} f(t_1, \dots, t_n) d\eta_\varepsilon(t_1) \dots d\eta_\varepsilon(t_n).
\end{aligned}$$

Therefore  $Y^\varepsilon(f)$  coincides with  $I_{n_\varepsilon}^3$  defined by (4).

Note also that in the case of multiple Wiener-Itô integrals ( $H = \frac{1}{2}$ ) the random variable  $Y^\varepsilon(f)_t$  coincides with  $I_{n_\varepsilon}(f)_t$  for  $\varepsilon$  small enough if  $f$  is a simple function.

### 3.1 The case $n = 2$

Let us consider first the case of a multiple integral in the second Wiener chaos. Suppose that  $f$  is a simple function of two variables of the form

$$f(x, y) = \sum_{k=1}^m \alpha_k 1_{(a_k^1, b_k^1]}(x) 1_{(a_k^2, b_k^2]}(y)$$

where for every  $k$ ,  $(a_k^1, b_k^1]$  and  $(a_k^2, b_k^2]$  are disjoint intervals. In this case, by using the product formula for multiple stochastic integrals (6), the multiple integral of  $f$  with respect to  $B^H$  can be expressed as

$$\begin{aligned} I_2^H(f 1_{[0,t]}^{\otimes 2}) &= \sum_{k=1}^m \alpha_k I_2^H \left( 1_{(a_k^1, b_k^1]} 1_{(a_k^2, b_k^2]} 1_{[0,t]}^{\otimes 2} \right) \\ &= \sum_{k=1}^m \alpha_k \left( B_{b_k^1 \wedge t}^H - B_{a_k^1 \wedge t}^H \right) \left( B_{b_k^2 \wedge t}^H - B_{a_k^2 \wedge t}^H \right) \\ &\quad - \sum_{k=1}^m \alpha_k \langle 1_{(a_k^1, b_k^1]} 1_{[0,t]}, 1_{(a_k^2, b_k^2]} 1_{[0,t]} \rangle_{\mathcal{H}}. \end{aligned} \quad (16)$$

The main difference with respect to the case of the standard Brownian motion is given by the fact that the scalar product in  $\mathcal{H}$  of two indicator functions of disjoint intervals is not zero anymore.

Let us show that the sequence

$$I_{2_\varepsilon}(f)_t = \int_0^1 \int_0^1 \left( \Gamma_H^{(2)} f 1_{[0,t]}^{\otimes 2} \right) (x_1, x_2) \left( \prod_{i=1}^2 \theta_\varepsilon(x_i) \right) 1_{\{|x_1 - x_2| > \varepsilon\}} dx_1 dx_2 \quad (17)$$

converges in the sense of finite dimensional distributions to the process  $I_2^H(f 1_{[0,t]}^{\otimes 2})$ . We can write

$$\begin{aligned} I_{2_\varepsilon}(f)_t &= \int_0^1 \int_0^1 \left( \Gamma_H^{(2)} f 1_{[0,t]}^{\otimes 2} \right) (x_1, x_2) \left( \prod_{i=1}^2 \theta_\varepsilon(x_i) \right) dx_1 dx_2 \\ &\quad - \int_0^1 \int_0^1 \left( \Gamma_H^{(2)} f 1_{[0,t]}^{\otimes 2} \right) (x_1, x_2) \left( \prod_{i=1}^2 \theta_\varepsilon(x_i) \right) 1_{\{|x_1 - x_2| < \varepsilon\}} dx_1 dx_2. \end{aligned} \quad (18)$$

Note that, using the properties of the transfer operator  $\Gamma_H^{(2)}$ , the first term can be written as

$$\begin{aligned} &\sum_k \alpha_k \int_0^1 \Gamma_H^{(1)} \left( 1_{(a_k^1, b_k^1] \cap [0,t]} \right) (x_1) \theta_\varepsilon(x_1) dx_1 \int_0^1 \Gamma_H^{(1)} \left( 1_{(a_k^2, b_k^2] \cap [0,t]} \right) (x_2) \theta_\varepsilon(x_2) dx_2 \\ &= \sum_k \alpha_k (\eta_\varepsilon(b_k^1 \wedge t) - \eta_\varepsilon(a_k^1 \wedge t)) (\eta_\varepsilon(b_k^2 \wedge t) - \eta_\varepsilon(a_k^2 \wedge t)) \end{aligned}$$

and by Lemma 2, its finite dimensional distributions converge to those of the stochastic process

$$\sum_{k=1}^m \alpha_k \left( B_{b_k^1 \wedge t}^H - B_{a_k^1 \wedge t}^H \right) \left( B_{b_k^2 \wedge t}^H - B_{a_k^2 \wedge t}^H \right).$$

Next we will discuss the behavior as  $\varepsilon \rightarrow 0$  of the second term. We need the following lemma, which will play an important role in the sequel.

**Lemma 3** Consider two functions  $f, g \in L^2([0, 1])$  and denote by

$$Y_\varepsilon = \int_0^1 \int_0^1 dx_1 dx_2 f(x_1) g(x_2) \theta_\varepsilon(x_1) \theta_\varepsilon(x_2) 1_{\{|x_1 - x_2| < \varepsilon\}}$$

where  $\theta_\varepsilon$  are the Kac-Stroock kernels or the Donsker kernels. Then

$$Y_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} Y = \int_0^1 f(x) g(x) dx \quad \text{in } L^2(\Omega).$$

**Proof:** The case when  $\theta_\varepsilon$  are the Kac-Stroock kernels. In this case,  $\theta_\varepsilon(x) = \frac{1}{\varepsilon}(-1)^{N(\frac{x}{\varepsilon})}$ , where  $\{N(t); t \geq 0\}$  is a standard Poisson process. We have

$$E(Y_\varepsilon - Y)^2 = E(Y_\varepsilon)^2 - 2YE(Y_\varepsilon) + E(Y)^2.$$

We first calculate,

$$\begin{aligned} E(Y_\varepsilon) &= \int_0^1 \int_0^1 dx_1 dx_2 f(x_1) g(x_2) \frac{1}{\varepsilon^2} e^{\frac{-2}{\varepsilon^2}|x_1 - x_2|} 1_{\{|x_1 - x_2| < \varepsilon\}} \\ &= \int_0^1 dx_1 f(x_1) \int_0^{x_1} dx_2 g(x_2) \frac{1}{\varepsilon^2} e^{\frac{-2}{\varepsilon^2}(x_1 - x_2)} 1_{\{0 < x_1 - x_2 < \varepsilon\}} \\ &\quad + \int_0^1 dx_2 g(x_2) \int_0^{x_2} dx_1 f(x_1) \frac{1}{\varepsilon^2} e^{\frac{-2}{\varepsilon^2}(x_2 - x_1)} 1_{\{0 < x_2 - x_1 < \varepsilon\}}. \end{aligned}$$

Note that

$$\int_0^{x_1} dx_2 g(x_2) \frac{2}{\varepsilon^2} e^{\frac{-2}{\varepsilon^2}(x_1 - x_2)} 1_{\{0 < x_1 - x_2 < \varepsilon\}} = g * \varphi_\varepsilon(x_1)$$

where  $\varphi_\varepsilon(z) = 1_{(0, \varepsilon)}(z) \frac{2}{\varepsilon^2} e^{-\frac{2z}{\varepsilon^2}}$  is an approximation of the identity. Therefore the convolution  $g * \varphi_\varepsilon$  converges to  $g$  in  $L^2([0, 1])$  because  $g \in L^2([0, 1])$ . We obtain

$$E(Y_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \int_0^1 f(x) g(x) dx = Y.$$

On the other hand,

$$\begin{aligned} &E((Y_\varepsilon)^2) \\ &= \int_{[0, 1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1) g(x_2) f(x_3) g(x_4) E(\theta_\varepsilon(x_1) \dots \theta_\varepsilon(x_4)) 1_{\{|x_1 - x_2| < \varepsilon\}} 1_{\{|x_3 - x_4| < \varepsilon\}} \\ &:= I_1^\varepsilon + I_2^\varepsilon \end{aligned}$$

with

$$\begin{aligned} I_1^\varepsilon &= \int_{[0, 1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1) g(x_2) f(x_3) g(x_4) \frac{1}{\varepsilon^4} e^{\frac{-2}{\varepsilon^2}(|x_2 - x_1|)} e^{\frac{-2}{\varepsilon^2}(|x_4 - x_3|)} \\ &\quad \times 1_{\{|x_2 - x_1| < \varepsilon\}} 1_{\{|x_4 - x_3| < \varepsilon\}} (1_{\{x_1 \vee x_2 < x_3 \wedge x_4\}} + 1_{\{x_3 \vee x_4 < x_1 \wedge x_2\}}) \end{aligned}$$

and

$$I_2^\varepsilon = \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) \frac{1}{\varepsilon^4} e^{\frac{-2}{\varepsilon^2}(x_2-x_1)} e^{\frac{-2}{\varepsilon^2}(x_4-x_3)} \\ \times 1_{\{|x_2-x_1|<\varepsilon\}} 1_{\{|x_4-x_3|<\varepsilon\}} 1_A(x_1, x_2, x_3, x_4)$$

where we denoted by  $A = \{\{x_1 \vee x_2 < x_3 \wedge x_4\} \cup \{x_3 \vee x_4 < x_1 \wedge x_2\}\}^C$ . We begin studying the convergence of the term  $I_2^\varepsilon$ . In the set  $A$ , there are 16 possible orders for the variables  $x_1, x_2, x_3, x_4$ . We will make the calculation for the case  $x_1 < x_3 < x_2 < x_4$  but for the other 15 possible orders we can proceed in a similar way. In this cas we have,

$$\int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) \frac{1}{\varepsilon^4} e^{\frac{-2}{\varepsilon^2}(x_3-x_1)} e^{\frac{-2}{\varepsilon^2}(x_4-x_2)} \\ \times 1_{\{x_1 < x_3 < x_2 < x_4\}} 1_{\{|x_2-x_1|<\varepsilon\}} 1_{\{|x_4-x_3|<\varepsilon\}} \\ \leq \frac{1}{2} \left( \int_{[0,1]^2} dx_1 dx_2 [f(x_1)g(x_2)]^2 1_{\{|x_2-x_1|<\varepsilon\}} \right. \\ \times \left( \int_{[0,1]^2} dx_3 dx_4 \frac{1}{\varepsilon^4} e^{\frac{-2}{\varepsilon^2}(x_3-x_1)} e^{\frac{-2}{\varepsilon^2}(x_4-x_2)} 1_{\{x_1 < x_3\}} 1_{\{x_2 < x_4\}} 1_{\{|x_4-x_3|<\varepsilon\}} \right) \\ + \int_{[0,1]^2} dx_3 dx_4 [f(x_3)g(x_4)]^2 1_{\{|x_4-x_3|<\varepsilon\}} \\ \times \left. \left( \int_{[0,1]^2} dx_1 dx_2 \frac{1}{\varepsilon^4} e^{\frac{-2}{\varepsilon^2}(x_3-x_1)} e^{\frac{-2}{\varepsilon^2}(x_4-x_2)} 1_{\{x_1 < x_3\}} 1_{\{x_2 < x_4\}} 1_{\{|x_2-x_1|<\varepsilon\}} \right) \right).$$

When we integrate the first integral with respect to  $dx_3 dx_4$  and the second integral with respect to  $dx_1 dx_2$  we obtain that the last expression can be bounded by

$$C \int_{[0,1]^2} dx_1 dx_2 [f(x_1)g(x_2)]^2 1_{\{|x_2-x_1|<\varepsilon\}}.$$

Proceeding in a similar way for the other 15 possible orders we obtain that

$$I_2^\varepsilon \leq C \int_{[0,1]^2} dx_1 dx_2 [f(x_1)g(x_2)]^2 1_{\{|x_2-x_1|<\varepsilon\}}.$$

This implies that  $I_2^\varepsilon$  converges to 0, by using the dominated convergence theorem.

Let us regard the behavior of the term  $I_1^\varepsilon$ . This term will give the convergence of  $E(Y^\varepsilon)^2$ . We have

$$I_1^\varepsilon = 8 \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) \frac{1}{\varepsilon^4} e^{\frac{-2}{\varepsilon^2}(|x_2-x_1|)} e^{\frac{-2}{\varepsilon^2}(|x_4-x_3|)} \\ \times 1_{\{|x_2-x_1|<\varepsilon\}} 1_{\{|x_4-x_3|<\varepsilon\}} 1_{\{x_1 < x_2 < x_3 < x_4\}} \\ = 2 \int_0^1 dx_2 g(x_2) 1_{\{x_2 < x_4\}} \int_0^{x_2} dx_1 f(x_1) \frac{2}{\varepsilon^2} e^{\frac{-2}{\varepsilon^2}(x_2-x_1)} 1_{\{0 < x_2-x_1 < \varepsilon\}} \\ \times \int_0^1 dx_4 g(x_4) \int_{x_2}^{x_4} dx_3 f(x_3) \frac{2}{\varepsilon^2} e^{\frac{-2}{\varepsilon^2}(x_4-x_3)} 1_{\{0 < x_4-x_3 < \varepsilon\}}.$$

We obtain that  $I_1^\varepsilon$  converges to  $2 \left[ \int_0^1 \int_0^1 f(x)g(x)f(y)g(y)1_{\{x < y\}} dx dy \right]$ . Thus  $I_1^\varepsilon$  converges to  $\left[ \int_0^1 f(x)g(x)dx \right]^2 = Y^2$ .

*The case when  $\theta_\varepsilon$  are the Donsker kernels.* In this case,  $\theta_\varepsilon(x) = \frac{1}{\varepsilon} \sum_{k=1}^\infty \xi_k 1_{[k-1, k)}(\frac{x}{\varepsilon^2})$  where  $(\xi_k)$  is a sequence of independent, identically distributed random variables satisfying  $E(\xi_1) = 0$  and  $E(\xi_1^2) = 1$  with  $E(\xi_1^{2n}) < +\infty$ . In this case we have

$$\begin{aligned} E(Y_\varepsilon) &= \int_0^1 \int_0^1 dx_1 dx_2 f(x_1)g(x_2) E(\theta_\varepsilon(x_1)\theta_\varepsilon(x_2)) 1_{\{|x_1 - x_2| < \varepsilon\}} \\ &= \frac{1}{\varepsilon^2} \sum_{k=1}^\infty \int_0^1 \int_0^1 dx_1 dx_2 f(x_1)g(x_2) 1_{[k-1, k)^2} \left( \frac{x_1}{\varepsilon^2}, \frac{x_2}{\varepsilon^2} \right) \\ &= \int_0^1 dx_1 f(x_1) \sum_{k=1}^\infty 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x_1) \frac{1}{\varepsilon^2} \int_{(k-1)\varepsilon^2}^{k\varepsilon^2} dx_2 g(x_2) \\ &= \int_0^1 dx_1 f(x_1) \sum_{k=1}^{\left[\frac{1}{\varepsilon^2}\right]+1} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x_1) \frac{1}{\varepsilon^2} \int_{(k-1)\varepsilon^2}^{k\varepsilon^2} dx_2 g(x_2) \\ &= \int_0^1 dx_1 f(x_1) G_\varepsilon(x_1), \end{aligned}$$

where

$$G_\varepsilon(x) := \sum_{k=1}^{\left[\frac{1}{\varepsilon^2}\right]+1} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x) \frac{1}{\varepsilon^2} \int_{(k-1)\varepsilon^2}^{k\varepsilon^2} dy g(y).$$

Fix  $x_1 \in (0, 1)$ . Then for every  $\varepsilon > 0$  close to zero, there exists an  $k(x_1, \varepsilon) \in \{1, \dots, \left[\frac{1}{\varepsilon^2}\right]+1\}$  such that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \leq x_1 < k(x_1, \varepsilon)\varepsilon^2$ . Then  $0 \leq x_1 - (k(x_1, \varepsilon) - 1)\varepsilon^2 < \varepsilon^2$  and this implies that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \rightarrow x_1$  as  $\varepsilon \rightarrow 0$ . Thus

$$G_\varepsilon(x_1) = \frac{1}{\varepsilon^2} \int_{(k(x_1, \varepsilon)-1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} dx_2 g(x_2)$$

converges to  $g(x_1)$  as  $\varepsilon \rightarrow 0$ . Consequently

$$E(Y_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \int_0^1 f(x_1)g(x_1)dx_1.$$

Now, we calculate  $E(Y_\varepsilon^2)$ . We have that,

$$\begin{aligned}
& E(\theta_\varepsilon(x_1) \dots \theta_\varepsilon(x_4)) \\
&= \frac{1}{\varepsilon^4} \sum_{k \neq j=1}^{\infty} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x_1, x_2) 1_{[(j-1)\varepsilon^2, j\varepsilon^2)}(x_3, x_4) \\
&\quad + \frac{1}{\varepsilon^4} \sum_{k \neq j=1}^{\infty} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x_1, x_3) 1_{[(j-1)\varepsilon^2, j\varepsilon^2)}(x_2, x_4) \\
&\quad + \frac{1}{\varepsilon^4} \sum_{k \neq j=1}^{\infty} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x_1, x_4) 1_{[(j-1)\varepsilon^2, j\varepsilon^2)}(x_2, x_3) \\
&\quad + \frac{E(\xi_1^4)}{\varepsilon^4} \sum_{k=1}^{\infty} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)}(x_1, x_2, x_3, x_4) \\
&:= G_\varepsilon^1(x_1, x_2, x_3, x_4) + G_\varepsilon^2(x_1, x_2, x_3, x_4) + G_\varepsilon^3(x_1, x_2, x_3, x_4) + G_\varepsilon^4(x_1, x_2, x_3, x_4).
\end{aligned}$$

Thus

$$\begin{aligned}
E((Y_\varepsilon)^2) &= \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^1(x_1, x_2, x_3, x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&\quad + \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^2(x_1, x_2, x_3, x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&\quad + \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^3(x_1, x_2, x_3, x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&\quad + \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 f(x_1)g(x_2)f(x_3)g(x_4) G_\varepsilon^4(x_1, x_2, x_3, x_4) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_3-x_4|<\varepsilon\}} \\
&:= J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3 + J_\varepsilon^4.
\end{aligned}$$

*The convergence of  $J_\varepsilon^1$ :* Fix  $x_1$  and  $x_3$  in  $[0, 1]$ . Then for every  $\varepsilon > 0$  close to zero, there exist  $k(x_1, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  and  $j(x_3, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  such that  $k(x_1, \varepsilon) \neq j(x_3, \varepsilon)$ ,  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \leq x_1 < k(x_1, \varepsilon)\varepsilon^2$  and  $(j(x_3, \varepsilon) - 1)\varepsilon^2 \leq x_3 < j(x_3, \varepsilon)\varepsilon^2$ , this implies that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \rightarrow x_1$  and  $(j(x_3, \varepsilon) - 1)\varepsilon^2 \rightarrow x_3$  as  $\varepsilon \rightarrow 0$ . Then we can write

$$\int_{[0,1]^2} dx_2 dx_4 g(x_2)g(x_4) G_\varepsilon^1(x_1, x_2, x_3, x_4) = \frac{1}{\varepsilon^2} \int_{(k(x_1, \varepsilon)-1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} dx_2 g(x_2) \times \frac{1}{\varepsilon^2} \int_{(j(x_3, \varepsilon)-1)\varepsilon^2}^{j(x_3, \varepsilon)\varepsilon^2} dx_4 g(x_4).$$

Moreover, this term converges to  $g(x_1)g(x_3)$ . We conclude that

$$J_\varepsilon^1 \xrightarrow{\varepsilon \rightarrow 0} \int_0^1 \int_0^1 dx_1 dx_3 f(x_1)g(x_1)f(x_3)g(x_3) = Y^2. \quad (19)$$

*The convergence of  $J_\varepsilon^2$  and  $J_\varepsilon^3$ :* Fix  $x_1$  and  $x_2$  in  $[0, 1]$ . Then for every  $\varepsilon > 0$  close to zero, there exist  $k(x_1, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  and  $j(x_2, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  such that

$k(x_1, \varepsilon) \neq j(x_2, \varepsilon)$ ,  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \leq x_1 < k(x_1, \varepsilon)\varepsilon^2$  and  $(j(x_2, \varepsilon) - 1)\varepsilon^2 \leq x_2 < j(x_2, \varepsilon)\varepsilon^2$ , this implies that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \rightarrow x_1$  and  $(j(x_2, \varepsilon) - 1)\varepsilon^2 \rightarrow x_2$  as  $\varepsilon \rightarrow 0$ . Hence

$$\begin{aligned} & \int_{[0,1]^2} dx_3 dx_4 g(x_3)g(x_4)G_\varepsilon^2(x_1, x_2, x_3, x_4)1_{\{|x_1-x_2|<\varepsilon\}}1_{\{|x_3-x_4|<\varepsilon\}} \\ & \leq 1_{\{|x_1-x_2|<\varepsilon\}} \frac{1}{\varepsilon^2} \int_{(k(x_1, \varepsilon)-1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} dx_3 g(x_3) \times \frac{1}{\varepsilon^2} \int_{(j(x_2, \varepsilon)-1)\varepsilon^2}^{j(x_2, \varepsilon)\varepsilon^2} dx_4 g(x_4). \end{aligned}$$

This last term converges to  $g(x_1)g(x_2)1_{\{x_1=x_2\}}$ , this implies that

$$J_\varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (20)$$

In the same way, we obtain that

$$J_\varepsilon^3 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (21)$$

*The convergence of  $J_\varepsilon^4$ :* Fix  $x_1$  in  $[0, 1]$ . Then for every  $\varepsilon > 0$  close to zero, there exist  $k(x_1, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  such that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \leq x_1 < k(x_1, \varepsilon)\varepsilon^2$ , this implies that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \rightarrow x_1$  as  $\varepsilon \rightarrow 0$ . Then we can write

$$\begin{aligned} & \int_{[0,1]^3} dx_2 dx_3 dx_4 g(x_2)f(x_3)g(x_4)G_\varepsilon^4(x_1, x_2, x_3, x_4) = \frac{E(\xi_1^4)}{\varepsilon^2} \int_{(k(x_1, \varepsilon)-1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} dx_2 g(x_2) \\ & \quad \times \frac{1}{\varepsilon^2} \int_{(k(x_1, \varepsilon)-1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} dx_3 f(x_3) \times \int_{(k(x_1, \varepsilon)-1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} dx_4 g(x_4). \end{aligned}$$

The last term converges to zero, thus

$$J_\varepsilon^4 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (22)$$

Consequently, by combining the above convergences we obtain that

$$E((Y_\varepsilon)^2) \xrightarrow{\varepsilon \rightarrow 0} Y^2. \quad (23)$$

■

We will also need the following lemma.

**Lemma 4** *Let us consider a family of stochastic processes  $(X^\varepsilon)_{t \in [0,1]}$  converging as  $\varepsilon \rightarrow 0$  to  $(X_t)_{t \in [0,1]}$  in the sense of finite dimensional distributions and a family of stochastic processes  $(Y^\varepsilon)_{t \in [0,1]}$  such that for every  $t \in [0, 1]$  the sequence of random variables  $Y_t^\varepsilon$  converges, as  $\varepsilon \rightarrow 0$  to  $Y_t$  in  $L^2(\Omega)$  where  $Y_t$  is a constant for every  $t$ . Then  $X^\varepsilon + Y^\varepsilon$  converges to  $X + Y$  in the sense of finite dimensional distributions.*

**Proof:** Fix  $t_1, \dots, t_r \in [0, 1]$  and let us show that the vector

$$(X_{t_1}^\varepsilon + Y_{t_1}^\varepsilon, \dots, X_{t_r}^\varepsilon + Y_{t_r}^\varepsilon)$$

converges in law to the vector

$$(X_{t_1} + Y_{t_1}, \dots, X_{t_r} + Y_{t_r}).$$

Take  $g \in \mathcal{C}_b^1(\mathbb{R}^r)$ , then

$$\begin{aligned} & |E(g(X_{t_1}^\varepsilon + Y_{t_1}^\varepsilon, \dots, X_{t_r}^\varepsilon + Y_{t_r}^\varepsilon)) - E(g(X_{t_1} + Y_{t_1}, \dots, X_{t_r} + Y_{t_r}))| \\ & \leq |E(g(X_{t_1}^\varepsilon + Y_{t_1}^\varepsilon, \dots, X_{t_r}^\varepsilon + Y_{t_r}^\varepsilon)) - E(g(X_{t_1}^\varepsilon + Y_{t_1}, \dots, X_{t_r}^\varepsilon + Y_{t_r}))| \\ & \quad + |E(g(X_{t_1}^\varepsilon + Y_{t_1}, \dots, X_{t_r}^\varepsilon + Y_{t_r})) - E(g(X_{t_1} + Y_{t_1}, \dots, X_{t_r} + Y_{t_r}))|. \end{aligned}$$

The first term converges to zero due to the  $L^2$  convergence of  $Y^\varepsilon$  to  $Y$  since

$$\begin{aligned} & |E(g(X_{t_1}^\varepsilon + Y_{t_1}^\varepsilon, \dots, X_{t_r}^\varepsilon + Y_{t_r}^\varepsilon)) - E(g(X_{t_1}^\varepsilon + Y_{t_1}, \dots, X_{t_r}^\varepsilon + Y_{t_r}))| \\ & \leq KE [(Y_{t_1}^\varepsilon - Y_{t_1})^2 + \dots + (Y_{t_r}^\varepsilon - Y_{t_r})^2]^{\frac{1}{2}} \end{aligned}$$

and the second one converges to zero as  $\varepsilon \rightarrow 0$  because, by Slutsky's theorem,  $X^\varepsilon + Y$  converges to  $X + Y$  in the sense of finite dimensional distributions.  $\blacksquare$

We obtain the following result:

**Proposition 1** *Let  $f \in \mathcal{S}_2$  and let  $I_{2\varepsilon}(f)_t$  be given by (17). Then  $(I_{2\varepsilon}(f)_t)_{t \in [0, 1]}$  converges as  $\varepsilon \rightarrow 0$  in the sense of finite dimensional distributions to the process  $(I_2^H(f 1_{[0, t]}^{\otimes 2}))_{t \in [0, 1]}$ .*

**Proof:** Recall the expressions (16) and (18) of  $I_2^H(f 1_{[0, t]}^{\otimes 2})$  and  $I_{2\varepsilon}(f)_t$ . By Lemma 2 the first term in (16) converges in the sense of finite dimensional distributions to the first term in (18) and applying Lemma 3 for  $f = \Gamma_H^{(1)} 1_{(a_k^1, b_k^1]} 1_{[0, t]}$  and  $g = \Gamma_H^{(1)} 1_{(a_k^2, b_k^2]} 1_{[0, t]}$  we obtain that the term

$$\int_0^1 \int_0^1 \left( \Gamma_H^{(2)} f 1_{[0, t]}^{\otimes 2} \right) (x_1, x_2) \left( \prod_{i=1}^2 \theta_\varepsilon(x_i) \right) 1_{\{|x_1 - x_2| < \varepsilon\}} dx_1 dx_2$$

converges in  $L^2(\Omega)$  for every  $t \in [0, 1]$  to

$$\sum_k \alpha_k \int_0^1 \Gamma_H^{(1)}(1_{(a_k^1, b_k^1]} 1_{[0, t]})(x) \Gamma_H^{(1)}(1_{(a_k^2, b_k^2]} 1_{[0, t]})(x) dx = \sum_k \alpha_k \langle 1_{(a_k^1, b_k^1]} 1_{[0, t]}, 1_{(a_k^2, b_k^2]} 1_{[0, t]} \rangle_{\mathcal{H}}.$$

The above Lemma 4 gives the conclusion.  $\blacksquare$



### 3.2 The case $n \geq 3$

In the case of multiple integrals of order  $n \geq 3$ , the structure of  $I_{n_\varepsilon}(f)_t$  is more complex because of the appearance of all diagonals. The first step is to express the multiple integral of a tensor product of one-variable functions.

**Lemma 5** *Let  $f_1, \dots, f_n \in \mathcal{H}$ . Then*

$$\begin{aligned} & I_n^H(f_1 \otimes f_2 \dots \otimes f_n) \\ &= \prod_{i=1}^n I_1^H(f_i) \\ &+ \sum_{l=1}^{[n/2]} (-1)^l \sum_{\substack{k_1, \dots, k_{2l} = 1; \\ k_j \text{ distinct}}}^n \left( \prod_{u \in \{1, \dots, n\} \setminus \{k_1, \dots, k_{2l}\}} I_1^H(f_u) \right) \langle f_{k_1}, f_{k_2} \rangle_{\mathcal{H}} \dots \langle f_{k_{2l-1}}, f_{k_{2l}} \rangle_{\mathcal{H}}. \end{aligned} \quad (24)$$

**Proof:** We will prove the result by induction. For  $n = 1, 2$  it is trivial. Let us show how it works for  $n = 3$  because it is useful to understand the general case. We have, using (10), (11) and the product formula for multiple integrals (6)

$$\begin{aligned} I_3^H(f_1 \otimes f_2 \otimes f_3) &= I_3 \left( \Gamma_H^{(3)}(f_1 \otimes f_2 \otimes f_3) \right) \\ &= I_3 \left( \Gamma_H^{(1)}(f_1) \otimes \Gamma_H^{(1)}(f_2) \otimes \Gamma_H^{(1)}(f_3) \right) \\ &= I_3 \left( \left( \Gamma_H^{(1)}(f_1) \otimes \Gamma_H^{(1)}(f_2) \otimes \Gamma_H^{(1)}(f_3) \right)^\sim \right) \\ &= I_2 \left( \Gamma_H^{(1)}(f_1) \tilde{\otimes} \Gamma_H^{(1)}(f_2) \right) I_1 \left( \Gamma_H^{(1)}(f_3) \right) \\ &\quad - 2I_1 \left( \left( \Gamma_H^{(1)}(f_1) \tilde{\otimes} \Gamma_H^{(1)}(f_2) \right) \otimes_1 \Gamma_H^{(1)}(f_3) \right). \end{aligned}$$

Note that

$$\left( \Gamma_H^{(1)}(f_1) \tilde{\otimes} \Gamma_H^{(1)}(f_2) \right) (t_1, t_2) = \frac{1}{2} \left( \Gamma_H^{(1)}(f_1)(t_1) \Gamma_H^{(1)}(f_2)(t_2) + \Gamma_H^{(1)}(f_1)(t_2) \Gamma_H^{(1)}(f_2)(t_1) \right)$$

and thus

$$\left( \Gamma_H^{(1)}(f_1) \tilde{\otimes} \Gamma_H^{(1)}(f_2) \right) \otimes_1 \Gamma_H^{(1)}(f_3) = \frac{1}{2} \left( \langle f_1, f_3 \rangle_{\mathcal{H}} \Gamma_H^{(1)}(f_2) + \langle f_2, f_3 \rangle_{\mathcal{H}} \Gamma_H^{(1)}(f_1) \right).$$

We obtain

$$\begin{aligned} I_3^H(f_1 \otimes f_2 \otimes f_3) &= \left( I_1^H(f_1) I_1^H(f_2) - \langle f_1, f_2 \rangle_{\mathcal{H}} \right) I_1^H(f_3) \\ &\quad + \left( \langle f_1, f_3 \rangle_{\mathcal{H}} I_1^H(f_2) + \langle f_2, f_3 \rangle_{\mathcal{H}} I_1^H(f_1) \right) \\ &= I_1^H(f_1) I_1^H(f_2) I_1^H(f_3) \\ &\quad - \left( \langle f_1, f_3 \rangle_{\mathcal{H}} I_1^H(f_2) + \langle f_2, f_3 \rangle_{\mathcal{H}} I_1^H(f_1) + \langle f_1, f_2 \rangle_{\mathcal{H}} I_1^H(f_3) \right). \end{aligned}$$

Concerning the general case, assume that (24) holds for  $1, 2, \dots, n-1$ . Again by the multiplication formula (6),

$$\begin{aligned}
& I_n^H(f_1 \otimes \dots \otimes f_n) \\
&= I_{n-1} \left( \Gamma_H^{(1)}(f_1) \otimes \dots \otimes \Gamma_H^{(1)}(f_{n-1}) \right) I_1 \left( \Gamma_H^{(1)}(f_n) \right) \\
&\quad - (n-1) I_{n-2} \left( \left( \Gamma_H^{(1)}(f_1) \otimes \dots \otimes \Gamma_H^{(1)}(f_{n-1}) \right)^\sim \otimes_1 \Gamma_H^{(1)}(f_n) \right) \\
&= I_1^H(f_n) \cdot \prod_{i=1}^{n-1} I_1^H(f_i) \\
&\quad + \sum_{l=1}^{[(n-1)/2]} (-1)^l \sum_{\substack{k_1, \dots, k_{2l} = 1; \\ k_j \text{ distinct}}}^{n-1} \left( \prod_{u \in \{1, \dots, n-1\} \setminus \{k_1, \dots, k_{2l}\}} I_1^H(f_u) I_1^H(f_n) \right) \langle f_{k_1}, f_{k_2} \rangle_{\mathcal{H}} \dots \langle f_{k_{2l-1}}, f_{k_{2l}} \rangle_{\mathcal{H}} \\
&\quad - \sum_{i=1}^{n-1} I_{n-2} \left( \left( \Gamma_H^{(1)}(f_1) \otimes \dots \otimes \hat{i} \dots \otimes \Gamma_H^{(1)}(f_{n-1}) \right)^\sim \right) \langle f_i, f_n \rangle_{\mathcal{H}}
\end{aligned}$$

and this equal to

$$\begin{aligned}
& I_1^H(f_n) \cdot \prod_{i=1}^{n-1} I_1^H(f_i) \\
&+ \sum_{l=1}^{[(n-1)/2]} (-1)^l \sum_{\substack{k_1, \dots, k_{2l} = 1; \\ k_j \text{ distinct}}}^{n-1} \left( \prod_{u \in \{1, \dots, n-1\} \setminus \{k_1, \dots, k_{2l}\}} I_1^H(f_u) I_1^H(f_n) \right) \langle f_{k_1}, f_{k_2} \rangle_{\mathcal{H}} \dots \langle f_{k_{2l-1}}, f_{k_{2l}} \rangle_{\mathcal{H}} \\
&- \sum_{i=1}^{n-1} \left( \prod_{j=1; j \neq i}^{n-1} I_1^H(f_j) \right) \\
&+ \sum_{l=1}^{[(n-2)/2]} (-1)^l \sum_{\substack{k_1, \dots, k_{2l} = 1; k_j \neq i; \\ k_j \text{ distinct}}}^{n-1} \left( \prod_{u \in \{1, \dots, \hat{i}, \dots, n-1\} \setminus \{k_1, \dots, k_{2l}\}} I_1^H(f_u) \right) \\
&\quad \times \langle f_{k_1}, f_{k_2} \rangle_{\mathcal{H}} \dots \langle f_{k_{2l-1}}, f_{k_{2l}} \rangle_{\mathcal{H}} \langle f_i, f_n \rangle_{\mathcal{H}}
\end{aligned}$$

and it is not difficult to see that the last quantity is equal to the right side of (24).  $\blacksquare$

The next auxiliary two lemmas will be used in the proof of the main result.

**Lemma 6** Suppose that  $(X_t^\varepsilon)_{t \in [0,1]}$  is a family of stochastic processes whose finite dimensional distributions converges to the finite dimensional distributions of a stochastic processes  $(X_t)_{t \geq 0}$ . Suppose also that  $(Y_{s,t}^\varepsilon)_{s,t \in [0,1]}$  is a two-parameter stochastic process such that for

every  $s, t \in [0, 1]$  we have that  $Y_{s,t}^\varepsilon$  converge in  $L^2(\Omega)$  to  $a_{s,t}$ , when  $\varepsilon$  tends to 0, where  $a_{s,t}$  is a real constant. Then for every  $t_1, \dots, t_{r_1} \in [0, 1]$  and  $s_1, \dots, s_{r_2}, u_1, \dots, u_{r_2} \in [0, 1]$  the vector

$$(X_{t_1}^\varepsilon, \dots, X_{t_{r_1}}^\varepsilon, Y_{s_1, u_1}^\varepsilon, \dots, Y_{s_{r_2}, u_{r_2}}^\varepsilon)$$

converges weakly to the vector

$$(X_{t_1}, \dots, X_{t_{r_1}}, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}}).$$

**Proof:** Consider  $f \in \mathcal{C}_b^1(\mathbb{R}^{r_1+r_2})$ . We have

$$\begin{aligned} & |E(f(X_{t_1}^\varepsilon, \dots, X_{t_{r_1}}^\varepsilon, Y_{s_1, u_1}^\varepsilon, \dots, Y_{s_{r_2}, u_{r_2}}^\varepsilon)) - E(f(X_{t_1}, \dots, X_{t_{r_1}}, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}}))| \\ & \leq |E(f(X_{t_1}^\varepsilon, \dots, X_{t_{r_1}}^\varepsilon, Y_{s_1, u_1}^\varepsilon, \dots, Y_{s_{r_2}, u_{r_2}}^\varepsilon)) - E(f(X_{t_1}^\varepsilon, \dots, X_{t_{r_1}}^\varepsilon, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}}))| \\ & \quad + |E(f(X_{t_1}^\varepsilon, \dots, X_{t_{r_1}}^\varepsilon, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}})) - E(f(X_{t_1}, \dots, X_{t_{r_1}}, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}}))| \\ & \leq KE \left[ (Y_{s_1, u_1}^\varepsilon - a_{s_1, u_1})^2 + \dots + (Y_{s_{r_2}, u_{r_2}}^\varepsilon - a_{s_{r_2}, u_{r_2}})^2 \right]^{\frac{1}{2}} \\ & \quad + |E(f(X_{t_1}^\varepsilon, \dots, X_{t_{r_1}}^\varepsilon, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}})) - E(f(X_{t_1}, \dots, X_{t_{r_1}}, a_{s_1, u_1}, \dots, a_{s_{r_2}, u_{r_2}}))|. \end{aligned}$$

The first term converges to zero because  $Y_{s,t}^\varepsilon$  converge in  $L^2(\Omega)$  to  $a_{s,t}$  and the second one converge to 0 because the finite dimensional distributions of  $(X_t^\varepsilon)_{t \in [0,1]}$  converge weakly to those of  $(X_t)_{t \geq 0}$ .  $\blacksquare$

**Lemma 7** Let  $f_i \in L^2([0, 1])$  for  $i = 1, \dots, n$  and define

$$F_\varepsilon = \int_{[0,1]^n} f_1(x_1) \cdots f_n(x_n) 1_{\{|x_1 - x_2| < \varepsilon\}} 1_{\{|x_1 - x_3| < \varepsilon\}} \left( \prod_{j=1}^n \theta_\varepsilon(x_j) \right) h_\varepsilon(x_1, \dots, x_n) dx_1 \cdots dx_n$$

where  $\theta_\varepsilon$  are the Kac-Stroock's or the Donsker's kernels and we assume that for every  $\varepsilon > 0$ ,  $|h_\varepsilon(x_1, \dots, x_n)| \leq 1$  for every  $x_1, \dots, x_n \in [0, 1]$ . Then

$$E(F_\varepsilon^2) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (25)$$

**Proof:** 1) Assume that  $\theta_\varepsilon$  are Kac-Stroock kernels. For  $\varepsilon > 0$  and  $x \geq 0$ , we set  $Q_\varepsilon(x) = \frac{1}{\varepsilon^2} e^{\frac{-2x}{\varepsilon^2}}$  and  $f_j := f_{j-n}$  for every  $j = n+1, \dots, 2n$ . We introduce some operators on the set of permutations.  $\mathfrak{S}_k$  denote the set of permutations on  $1, \dots, k$ . When  $\tau \in \mathfrak{S}_{2m}$  and  $\sigma \in \mathfrak{S}_m$ , we note  $\sigma \star \tau$  the element of  $\mathfrak{S}_{2m}$  defined by

$$\sigma \star \tau(2j-1) = \tau(2\sigma(j)-1) \text{ and } \sigma \star \tau(2j) = \tau(2\sigma(j)).$$

We have  $id \star \tau = \tau$  and  $\sigma' \star (\sigma \star \tau) = (\sigma' \sigma) \star \tau$ , hence  $\star : \mathfrak{S}_m \times \mathfrak{S}_{2m} \rightarrow \mathfrak{S}_{2m}$  defines a (right) group action of  $\mathfrak{S}_m$  on  $\mathfrak{S}_{2m}$ . For any  $\tau \in \mathfrak{S}_{2m}$ , the orbit of  $\tau$  has exactly  $m!$

elements. Consequently, the set  $\mathcal{O}$  of the orbits under the group action  $\star$  has  $\frac{(2m)!}{m!}$  elements and we have, by denoting  $\tau_i$  one particular element of the orbit  $o_i = o(\tau_i) \in \mathcal{O}$  : for  $r_1, \dots, r_{2m} \in [0, 1]$ ,

$$\begin{aligned} 1_{\{\forall i \neq j, r_i \neq r_j\}} &= \sum_{\tau \in \mathfrak{S}_{2m}} 1_{\{r_{\tau(1)} > \dots > r_{\tau(2m)}\}} = \sum_{o_i \in \mathcal{O}} \sum_{\tau \in o_i} 1_{\{r_{\tau(1)} > \dots > r_{\tau(2m)}\}} \\ &\leq \prod_{i=1}^{\frac{(2m)!}{m!}} \prod_{j=1}^m 1_{\{r_{2\tau_i(j)-1} > r_{2\tau_i(j)}\}}. \end{aligned} \quad (26)$$

Then for any  $\varepsilon > 0$ , we have

$$\begin{aligned} E(F_\varepsilon^2) &= \left| \int_{[0,1]^{2n}} f_1(x_1) \dots f_{2n}(x_{2n}) 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} 1_{\{|x_{n+1}-x_{n+2}|<\varepsilon\}} 1_{\{|x_{n+1}-x_{n+3}|<\varepsilon\}} \right. \\ &\quad \times E \left( \prod_{j=1}^n \theta_\varepsilon(x_j) \theta_\varepsilon(x_{j+1}) \right) h_\varepsilon(x_1, \dots, x_n) h_\varepsilon(x_{j+1}, \dots, x_{2n}) dx_1 \dots dx_{2n} \Big| \\ &\leq \int_{[0,1]^{2n}} |f_1(x_1)| \dots |f_{2n}(x_{2n})| 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} \left| E \left( \prod_{j=1}^{2n} \theta_\varepsilon(x_j) \right) \right| dx_1 \dots dx_{2n} \\ &= \sum_{o_i \in \mathcal{O}} \sum_{\tau \in o_i} \int_{[0,1]^{2n}} |f_1(x_1)| \dots |f_{2n}(x_{2n})| 1_{\{|x_1-x_2|<\varepsilon\}} 1_{\{|x_1-x_3|<\varepsilon\}} \\ &\quad \times 1_{\{x_{\tau(1)} > \dots > x_{\tau(2n)}\}} Q_\varepsilon \left( \sum_{j=1}^n (x_{\tau(2j-1)} - x_{\tau(2j)}) \right) dx_1 \dots dx_{2n}. \end{aligned}$$

Among the addends of the last term there are two possible situations.

- On one hand we have terms of the type:

$$\begin{aligned} &\int_{[0,1]^4} 1_{\{x_3 > x_4\}} |f_{2\tau_i(k)-1}(x_1)| |f_{2\tau_i(k)}(x_2)| Q_\varepsilon(x_1 - x_2) \\ &\times 1_{\{0 < x_1 - x_2 < \varepsilon\}} 1_{\{0 < x_1 - x_3 < \varepsilon\}} |f_{2\tau_i(k')-1}(x_3)| |f_{2\tau_i(k')}(x_4)| Q_\varepsilon(x_3 - x_4) dx_1 dx_2 dx_3 dx_4 \\ &\times \prod_{j \neq k, k';=1}^n \int_{[0,1]^2} 1_{\{x_1 > x_2\}} |f_{2\tau_i(j)-1}(x_1)| |f_{2\tau_i(j)}(x_2)| Q_\varepsilon(x_1 - x_2) dx_1 dx_2, \end{aligned}$$

where  $\tau_i(k) > \tau_i(k') + 1$ .

Notice that, using (26)(as in [5]), we obtain

$$\begin{aligned} &\int_{[0,1]^2} 1_{\{x_1 > x_2\}} |f_{2\tau_i(j)-1}(x_1)| |f_{2\tau_i(j)}(x_2)| Q_\varepsilon(x_1 - x_2) dx_1 dx_2 \\ &\leq \frac{1}{2} \|f_{2\tau_i(j)-1}\|_{L^2} \|f_{2\tau_i(j)}\|_{L^2}. \end{aligned}$$

Moreover, given  $h_i \in L^2([0, 1])$ ,  $i = 1, \dots, 4$ , we have that

$$\begin{aligned}
& \int_{[0,1]^4} |h_1(x_1)| |h_2(x_2)| |h_3(x_3)| |h_4(x_4)| 1_{\{0 < x_1 - x_2 < \varepsilon\}} 1_{\{0 < x_1 - x_3 < \varepsilon\}} \\
& \times Q_\varepsilon(x_1 - x_2) Q_\varepsilon(x_3 - x_4) dx_1 dx_2 dx_3 dx_4 \\
\leq & \int_{[0,1]^4} |h_1(x_1)| |h_2(x_2)| |h_3(x_3)| |h_4(x_4)| 1_{\{0 < x_1 - x_2 < \varepsilon\}} 1_{\{0 < x_1 - x_3 < \varepsilon\}} 1_{\{0 < x_3 - x_4 < \varepsilon\}} \\
& \times Q_\varepsilon(x_1 - x_2) Q_\varepsilon(x_3 - x_4) dx_1 dx_2 dx_3 dx_4 \\
& + \int_{[0,1]^4} |h_1(x_1)| |h_2(x_2)| |h_3(x_3)| |h_4(x_4)| 1_{\{0 < x_1 - x_2 < \varepsilon\}} 1_{\{0 < x_1 - x_3 < \varepsilon\}} 1_{\{\varepsilon < x_3 - x_4\}} \\
& \times Q_\varepsilon(x_1 - x_2) Q_\varepsilon(x_3 - x_4) dx_1 dx_2 dx_3 dx_4 \\
:= & A_\varepsilon^1 + A_\varepsilon^2.
\end{aligned}$$

The term  $A_\varepsilon^1$  converges to zero by using the same manner of the convergence of  $I_2^\varepsilon$  in the proof of Lemma 3. For the term  $A_\varepsilon^2$  we have that

$$\begin{aligned}
& A_\varepsilon^2 \\
\leq & 2 \left( \int_{[0,1]^4} h_1^2(x_1) h_3^2(x_3) 1_{\{0 < x_1 - x_3 < \varepsilon\}} Q_\varepsilon(x_1 - x_2) Q_\varepsilon(x_3 - x_4) dx_1 dx_2 dx_3 dx_4 \right. \\
& \left. + \int_{[0,1]^4} h_2^2(x_2) h_4^2(x_4) 1_{\{0 < x_1 - x_3 < \varepsilon\}} 1_{\{\varepsilon < x_3 - x_4\}} Q_\varepsilon(x_1 - x_2) Q_\varepsilon(x_3 - x_4) dx_1 dx_2 dx_3 dx_4 \right).
\end{aligned}$$

Integrating with respect to  $x_2$  and  $x_4$  in the first addend we obtain the convergence to zero by using the dominated convergence theorem. Moreover, using the fact that for  $y > \varepsilon$ ,  $Q_\varepsilon(y) \leq e^{-2}$ , and integrating after with respect to  $x_3$  and  $x_1$  we can bound the second addend by

$$\varepsilon \int_{[0,1]^2} h_2^2(x_2) h_4^2(x_4) dx_2 dx_4$$

that clearly converge also to zero.

- We have also terms of the type:

$$\begin{aligned}
& \int_{[0,1]^6} 1_{\{x_1 > x_4\}} |f_{2\tau_i(k)-1}(x_1)| |f_{2\tau_i(k)}(x_4)| Q_\varepsilon(x_1 - x_4) \\
& \times 1_{\{x_2 > x_5\}} |f_{2\tau_i(k')-1}(x_2)| |f_{2\tau_i(k')}(x_5)| Q_\varepsilon(x_2 - x_5) \\
& \times 1_{\{x_3 > x_6\}} |f_{2\tau_i(k'')-1}(x_3)| |f_{2\tau_i(k'')}(x_6)| Q_\varepsilon(x_3 - x_6) \\
& \times 1_{\{0 < x_1 - x_2 < \varepsilon\}} 1_{\{0 < x_1 - x_3 < \varepsilon\}} dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \\
& \times \prod_{j \neq k, k', k''; =1}^n \int_{[0,1]^2} 1_{\{x_1 > x_2\}} |f_{2\tau_i(j)-1}(x_1)| |f_{2\tau_i(j)}(x_2)| Q_\varepsilon(x_1 - x_2) dx_1 dx_2,
\end{aligned}$$

where  $\tau_i(k) > \tau_i(k') + 1 > \tau_i(k'') + 2$ .

But, using arguments similar to those presented in the previous situation it is not difficult to see that also this type of terms converges to zero.

Combining the above convergences we conclude that  $E(F_\varepsilon^2)$  converges to zero and thus the Lemma 7 satisfied.

2) Assume now that  $\theta_\varepsilon$  are Donsker kernels. For any  $m \geq 3$

$$G_{\varepsilon,m}(x_1, \dots, x_m) := \frac{E(\xi_1^m)}{\varepsilon^m} \sum_{k=1}^{\infty} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)^m}(x_1, \dots, x_m).$$

Fix  $x_1$  in  $[0, 1]$ . Then for every  $\varepsilon > 0$  close to zero, there exist  $k(x_1, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  such that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \leq x_1 < k(x_1, \varepsilon)\varepsilon^2$ , this implies that  $(k(x_1, \varepsilon) - 1)\varepsilon^2 \rightarrow x_1$  as  $\varepsilon \rightarrow 0$ . Then we can write

$$\begin{aligned} J_{\varepsilon,m} &:= \int_{[0,1]^m} f_1(x_1) f_2(x_2) \dots f_m(x_m) G_{\varepsilon,m}(x_1, \dots, x_m) dx_1 \dots dx_m \\ &= E(\xi_1^m) \varepsilon^{m-2} \int_0^1 f_1(x_1) \prod_{j=2}^m \left[ \frac{1}{\varepsilon^2} \int_{(k(x_1, \varepsilon) - 1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} f_j(x_j) dx_j \right] dx_1. \end{aligned}$$

Moreover for each  $j = 2, \dots, m$ , the term  $\frac{1}{\varepsilon^2} \int_{(k(x_1, \varepsilon) - 1)\varepsilon^2}^{k(x_1, \varepsilon)\varepsilon^2} f_j(x_j) dx_j$  converges to  $f_j(x_1)$ . Combining this with  $m \geq 3$ , we obtain that

$$J_{\varepsilon,m} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (27)$$

On the other hand, if we denote by

$$\bar{G}_{\sigma,\varepsilon}(x_1, \dots, x_{2n}) = \frac{1}{\varepsilon^{2n}} \prod_{j=0}^{n-1} \left( \sum_{k=1}^{\infty} 1_{[(k-1)\varepsilon^2, k\varepsilon^2)^2}(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) \right)$$

Fix  $x_{\sigma(2j+1)} \in [0, 1]$  for any  $j = 0, \dots, n-1$ . Then for every  $\varepsilon > 0$  close to zero, there exist  $k(x_{\sigma(2j+1)}, \varepsilon) \in \{1, \dots, [\frac{1}{\varepsilon^2}] + 1\}$  such that

$$k(x_{\sigma(2j+1)}, \varepsilon) \neq k(x_{\sigma(2j'+1)}, \varepsilon) \quad \forall j' \neq j \text{ and } (k(x_{\sigma(2j+1)}, \varepsilon) - 1)\varepsilon^2 \leq x_{\sigma(2j+1)} < k(x_{\sigma(2j+1)}, \varepsilon)\varepsilon^2$$

this implies that  $(k(x_{\sigma(2j+1)}, \varepsilon) - 1)\varepsilon^2 \rightarrow x_{\sigma(2j+1)}$  as  $\varepsilon \rightarrow 0$ . Then we can write

$$\begin{aligned} \bar{J}_{\sigma,\varepsilon} &:= \int_{[0,1]^{2n}} f_1(x_1) \dots f_{2n}(x_{2n}) \bar{G}_{\sigma,\varepsilon}(x_1, \dots, x_{2n}) 1_{\{|x_1 - x_2| < \varepsilon\}} 1_{\{|x_1 - x_3| < \varepsilon\}} dx_1 \dots dx_{2n} \\ &\leq \int_{[0,1]^n} \prod_{l=0}^{n-1} \left[ f_{\sigma(2j+1)}(x_{\sigma(2j+1)}) \frac{1}{\varepsilon^2} \int_{(k(x_{\sigma(2j+1)}, \varepsilon) - 1)\varepsilon^2}^{k(x_{\sigma(2j+1)}, \varepsilon)\varepsilon^2} 1_{\{|x_1 - x_2| < \varepsilon\}} 1_{\{|x_1 - x_3| < \varepsilon\}} \right. \\ &\quad \times \left. f_{\sigma(2j+2)}(x_{\sigma(2j+2)}) dx_{\sigma(2j+2)} \right] dx_{\sigma(1)} dx_{\sigma(3)} \dots dx_{\sigma(2n-1)}. \end{aligned}$$

Moreover, this last term converges to

$$\begin{aligned} & \int_{[0,1]^n} 1_{\{x_{\sigma(2k+1)}=x_{\sigma(2k'+1)}; \text{ for some } k \neq k'\}} \\ & \times \prod_{l=0}^{n-1} [f_{\sigma(2l+1)}(x_{\sigma(2l+1)}) f_{\sigma(2l+2)}(x_{\sigma(2l+1)})] dx_{\sigma(1)} dx_{\sigma(3)} \dots dx_{\sigma(2n-1)} = 0. \end{aligned} \quad (28)$$

From (27), (28) and the fact that the term  $E\left(\prod_{j=1}^n \theta_\varepsilon(x_j)\right)$  is written as a sum of terms of type  $G_{\varepsilon,m}$  or  $\bar{G}_{\sigma,\varepsilon}$  we conclude that

$$E(F_\varepsilon^2) \xrightarrow{\varepsilon \rightarrow 0} 0$$

■

We can state now our approximation result for multiple fractional integrals when the integrand is a simple function.

**Proposition 2** *Let  $f$  be a simple function of the form (15). Then the finite dimensional distribution of the process (14) converge as  $\varepsilon \rightarrow 0$  to the finite dimensional distributions of  $\left(I_n^H(f 1_{[0,t]}^{\otimes n})\right)_{t \in [0,1]}$ .*

**Proof:** If  $f$  is a simple function of the form (15) then for every  $t \in [0, 1]$

$$\begin{aligned} I_n^H(f 1_{[0,t]}^{\otimes n}) &= \sum_{k=1}^m \alpha_k I_n^H \left( 1_{(a_k^1, b_k^1]} \times \dots \times 1_{(a_k^n, b_k^n]} 1_{[0,t]}^{\otimes n} \right) \\ &= \sum_{k=1}^m \alpha_k \left( \prod_{i=1}^n I_1^H(1_{(a_k^i, b_k^i]} 1_{[0,t]}) \right) \\ &\quad + \sum_{l=1}^{[n/2]} (-1)^l \sum_{\substack{j_1, \dots, j_{2l} = 1; \\ j_i \text{ distinct}}}^n \left( \prod_{u \in \{1, \dots, n\} \setminus \{j_1, \dots, j_{2l}\}} I_1^H \left( 1_{(a_k^u, b_k^u]} 1_{[0,t]} \right) \right) \\ &\quad \times \langle 1_{(a_k^{j_1}, b_k^{j_1]} 1_{[0,t]}, 1_{(a_k^{j_2}, b_k^{j_2]} 1_{[0,t]}} \rangle \mathcal{H} \dots \langle 1_{(a_k^{j_{2l-1}}, b_k^{j_{2l-1}}} 1_{[0,t]}, 1_{(a_k^{j_{2l}}, b_k^{j_{2l}}} 1_{[0,t]}} \rangle \mathcal{H} \rangle. \end{aligned}$$

The approximation  $I_{n_\varepsilon}(f)_t$  can be expressed as

$$\begin{aligned}
I_{n_\varepsilon}(f)_t &= \int_{[0,1]^n} \left( \Gamma_H^{(n)} f 1_{[0,t]}^n \right) (x_1, \dots, x_n) \prod_{j=1}^n \theta_\varepsilon(x_j) \left( \prod_{i,j=1; i \neq j}^n (1 - 1_{\{|x_i - x_j| < \varepsilon\}}) \right) dx_1 \dots dx_n \\
&= \int_{[0,1]^n} \left( \Gamma_H^{(n)} f 1_{[0,t]}^n \right) (x_1, \dots, x_n) \prod_{j=1}^n \theta_\varepsilon(x_j) \\
&\quad \times \left( 1 + \sum_{l=1}^{\lfloor n/2 \rfloor} (-1)^l \left( \sum_{\substack{k_1, \dots, k_{2l} = 1; \\ k_j \text{ distinct}}}^n 1_{\{|x_{k_1} - x_{k_2}| < \varepsilon\}} \cdots 1_{\{|x_{k_{2l-1}} - x_{k_{2l}}| < \varepsilon\}} dx_1 \cdots dx_n \right) \right) + R.
\end{aligned}$$

The term  $R$  above contains terms of the type

$$\int_{[0,1]^n} \left( \prod_{j=1}^n (\Gamma_H^{(1)} 1_{(a_k^j, b_k^j]} 1_{[0,t]}) (x_j) \theta_\varepsilon(x_j) \right) 1_{\{|x_1 - x_2| < \varepsilon\}} 1_{\{|x_1 - x_3| < \varepsilon\}} 1_A(x_1, \dots, x_n)$$

where  $A$  is a Borel subset of  $[0,1]^{\otimes n}$ . It will converge to zero by using Lemma 7 for  $h_\varepsilon(x_1, \dots, x_n) = 1_A(x_1, \dots, x_n)$ . The behavior of  $I_{n_\varepsilon}(f)_t$  will be then given by the behavior of

$$\begin{aligned}
&\int_{[0,1]^n} \left( \Gamma_H^{(n)} f 1_{[0,t]}^n \right) (x_1, \dots, x_n) \prod_{j=1}^n \theta_\varepsilon(x_j) \\
&\times \left( 1 + \sum_{l=1}^{\lfloor n/2 \rfloor} (-1)^l \left( \sum_{\substack{k_1, \dots, k_{2l} = 1; \\ k_j \text{ distinct}}}^n 1_{\{|x_{k_1} - x_{k_2}| < \varepsilon\}} \cdots 1_{\{|x_{k_{2l-1}} - x_{k_{2l}}| < \varepsilon\}} dx_1 \cdots dx_n \right) \right).
\end{aligned}$$

First we note that by Lemma 2 the first term in the above expression converges in the sense of finite dimensional distributions to

$$\sum_k \alpha_k I_1^H(1_{(a_k^1, b_k^1]} 1_{[0,t]}) \cdots I_1^H(1_{(a_k^n, b_k^n]} 1_{[0,t]}) = \sum_k \alpha_k (B_{b_k^1 \wedge t}^H - B_{a_k^1 \wedge t}^H) \cdots (B_{b_k^n \wedge t}^H - B_{a_k^n \wedge t}^H).$$

We will show that for every  $l = 1, \dots, \lfloor \frac{n}{2} \rfloor$  and for every  $j_1, \dots, j_{2l} = 1, \dots, n$  distinct the sequence

$$\int_{[0,1]^n} dx_1 \cdots dx_n \Gamma_H^{(n)}(f 1_{[0,t]}^{\otimes n})(x_1, \dots, x_n) \left( \prod_{j=1}^n \theta_\varepsilon(x_j) \right) 1_{\{|x_{j_1} - x_{j_2}| < \varepsilon\}} \cdots 1_{\{|x_{j_{2l-1}} - x_{j_{2l}}| < \varepsilon\}}$$



converges in the sense of finite dimensional distributions to the stochastic process

$$\sum_k \alpha_k \left( \prod_{u \in \{1, \dots, n\} \setminus \{j_1, \dots, j_{2l}\}} I_1^H \left( 1_{(a_k^u, b_k^u]} 1_{[0, t]} \right) \right) \\ \times \langle 1_{(a_k^{j_1}, b_k^{j_1}]} 1_{[0, t]}, 1_{(a_k^{j_2}, b_k^{j_2}]} 1_{[0, t]} \rangle \mathcal{H} \cdots \langle 1_{(a_k^{j_{2l-1}}, b_k^{j_{2l-1}}]} 1_{[0, t]}, 1_{(a_k^{j_{2l}}, b_k^{j_{2l}}]} 1_{[0, t]} \rangle \mathcal{H}.$$

Indeed, since

$$\Gamma_H^{(n)}(f 1_{[0, t]}^{\otimes n})(x_1, \dots, x_n) = \sum_{k=1}^m \alpha_k (\Gamma_H^{(1)} 1_{(a_k^1, b_k^1]})(x_1) \cdots (\Gamma_H^{(1)} 1_{(a_k^n, b_k^n]})(x_n)$$

we can write, for every  $j_1, \dots, j_{2l} = 1, \dots, n$  distinct

$$\int_{[0, 1]^n} dx_1 \cdots dx_n \Gamma_H^{(n)}(f 1_{[0, t]}^{\otimes n})(x_1, \dots, x_n) \left( \prod_{j=1}^n \theta_\varepsilon(x_j) \right) 1_{\{|x_{j_1} - x_{j_2}| < \varepsilon\}} \cdots 1_{\{|x_{j_{2l-1}} - x_{j_{2l}}| < \varepsilon\}} \\ = \sum_{k=1}^m \alpha_k \left( \prod_{u \in \{1, \dots, n\} \setminus \{j_1, \dots, j_{2l}\}} \int_{[0, 1]} dx_u (\Gamma_H^{(1)} 1_{(a_k^u, b_k^u]} 1_{[0, t]})(x_u) \theta_\varepsilon(x_u) \right) \\ \times \int_{[0, 1]^2} (\Gamma_H^{(1)} 1_{(a_k^{j_1}, b_k^{j_1}]} 1_{[0, t]})(x_{j_1}) (\Gamma_H^{(1)} 1_{(a_k^{j_2}, b_k^{j_2}]} 1_{[0, t]})(x_{j_2}) \theta_\varepsilon(x_{j_1}) \theta_\varepsilon(x_{j_2}) 1_{\{|x_{j_1} - x_{j_2}| < \varepsilon\}} dx_{j_1} dx_{j_2} \\ \cdots \times \int_{[0, 1]^2} (\Gamma_H^{(1)} 1_{(a_k^{j_{2l-1}}, b_k^{j_{2l-1}}]} 1_{[0, t]})(x_{j_{2l-1}}) (\Gamma_H^{(1)} 1_{(a_k^{j_{2l}}, b_k^{j_{2l}}]} 1_{[0, t]})(x_{j_{2l}}) \\ \times \theta_\varepsilon(x_{j_{2l-1}}) \theta_\varepsilon(x_{j_{2l}}) 1_{\{|x_{j_{2l-1}} - x_{j_{2l}}| < \varepsilon\}} dx_{j_{2l-1}} dx_{j_{2l}} \\ = \sum_{k=1}^m \alpha_k \left( \prod_{u \in \{1, \dots, n\} \setminus \{j_1, \dots, j_{2l}\}} (\eta_\varepsilon(b_k^u \wedge t) - \eta_\varepsilon(a_k^u \wedge t)) \right) \\ \times \left( Y_{b_k^{j_1} \wedge t, b_k^{j_2} \wedge t}^\varepsilon - Y_{a_k^{j_1} \wedge t, b_k^{j_2} \wedge t}^\varepsilon - Y_{b_k^{j_1} \wedge t, a_k^{j_2} \wedge t}^\varepsilon + Y_{a_k^{j_1} \wedge t, a_k^{j_2} \wedge t}^\varepsilon \right)$$

where, for  $v = 1, \dots, l$  we denoted by

$$Y_{s, t}^\varepsilon = \int_{[0, 1]^2} (\Gamma_H^{(1)} 1_{[0, t]})(x_1) (\Gamma_H^{(1)} 1_{[0, s]})(x_2) \theta_\varepsilon(x_1) \theta_\varepsilon(x_2) 1_{\{|x_1 - x_2| < \varepsilon\}} dx_1 dx_2.$$

The conclusion follows by using Lemma 6 and the results obtained for the case  $n = 2$ .  $\blacksquare$

We state now our main result.

**Theorem 1** *Let  $f$  be a function in the space  $|\mathcal{H}|^{\otimes n}$ . Then the finite dimensional distribution of the process (14) converge as  $\varepsilon \rightarrow 0$  to the finite dimensional distributions of  $(I_n^H(f 1_{[0, t]}^{\otimes n}))_{t \in [0, 1]}$ .*

**Proof:** It is a consequence of Lemma 2.1 and Theorem 2.3 in [4], of the isometry of multiple integrals and of the fact that the simple functions are dense in  $|\mathcal{H}|^{\otimes n}$  since for every  $t \in [0, 1]$  it holds (see Section 2.2 in [4])

$$E |I_{n_\varepsilon}(f)_t| \leq c \|\Gamma_H^{(n)} f 1_{[0,t]}^{\otimes n}\|_{L^2([0,1]^{\otimes n})} = c \|f 1_{[0,t]}^{\otimes n}\|_{\mathcal{H}^{\otimes n}}.$$

■

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